European Economic Review 21 (1983) 147–166. North-Holland

ON THE IMPACT OF SHOCK PERSISTENCE ON THE DYNAMICS OF A RECURSIVE ECONOMY*

Jean-Pierre DANTHINE

University of Lausanne, CH-1015 Lausanne, Switzerland

John B. DONALDSON and Rajnish MEHRA

Columbia University, New York, NY 10027, USA

Received September 1981, final version received September 1982

The present note relies exclusively on numerical computation of a parametric version of a (stochastic) version of the one-sector neoclassical growth model to derive the qualitative properties of the optimal consumption/investment policy functions and of the resultant steady state, and to study the manner in which these properties are affected by an increase in the degree of shock persistence. In particular, we measure the effects of (differing degrees of) shock persistence on the means and variances of the resulting (stationary) distributions on output, consumption, and capital stock. Furthermore, we explore the effects of increasing degrees of shock persistence on the dynamic time path of the economy.

1. Introduction

Considerable interest has been generated by the extension to uncertainty of the theory of optimal growth. This extension culminated with Brock and Mirman's (1972) proof that the stochastic version of the one-sector neoclassical model with concave technology behaves analogously to the deterministic formulation of Cass (1965) and Koopmans (1965). That is, optimal savings (investment) policies exist and by their application the economy converges (in an appropriate stochastic sense) to a steady state distribution. The attractiveness of this development was further enhanced by the correspondence, established by Brock (1978) and Prescott and Mehra (1980), between the optimal policies of this model and the aggregate

*Danthine gratefully acknowledges financial support from the Fonds National de la Recherche Scientifique (Switzerland). Donaldson and Mehra's work was supported by the National Science Foundation and by the Faculty Research Fund of the Graduate School of Business, Columbia University. In addition, Mehra would like to thank Ecole des H.E.C., Université de Lausanne, for its hospitality during the initial stage of this research. We thank Jean Laville for very efficiently performing all the computations reported in this note.

0014–2921/83/$3.00

equilibrium policies of an appropriately specified homogeneous agent decentralized economy.

All studies employing this model [e.g. Brock and Mirman (1972) and Brock (1978)] have thus far assumed that the random shocks affecting the technology were independently and identically distributed through time. This was not, however, an expression of belief in the theoretical appropriateness of this assumption. Rather, the i.i.d. assumption was adopted only to facilitate analytic tractability, while providing a natural first approximation to reality as well. It is clear however that this state of affairs is transitory and that the implications of non-zero correlation in shocks need to be worked out. Indeed, whether the random elements are interpreted as capturing observational errors in the measurement of aggregate capital stock, actual uncertainty in the production process (e.g. as crop yield is influenced by uncertain rainfall) or psychological characteristics such as optimism or confidence in the future, some degree of autocorrelation is likely to be present in the corresponding real world phenomenon.

The introduction of (positive) correlation in the random shocks forces a reexamination of this theory at different levels. Donaldson and Mehra (1983) demonstrate the existence of optimal consumption/investment policy functions and show that by their application the economy converges to a well defined steady state. The present note relies exclusively on numerical computation of a parametric version of this model to derive the qualitative properties of these policy functions and the resultant steady state, and to study the manner in which these properties are affected by an increase in the degree of shock persistence. In particular, we measure the effects of (differing degrees of) shock persistence on the means and variances of the resulting (stationary) distributions on output, consumption, and capital stock. Furthermore, we explore the effects of increasing degrees of shock persistence on the dynamic time path of the economy.

In doing so we seek to give an indication of the extent to which a simple and raw model such as this one — incorporating neither the information structure of Lucas (1975), nor the 'time to build' assumption of Kydland and Prescott (1982), nor the multigood structure of Long and Plosser (1980) — can qualitatively approximate the observed behavior of macroaggregates of output, consumption, and capital stock. Measures characterizing the dynamics of these macroseries are recorded (e.g. autocorrelation of output) and graphical illustrations are provided. Again, the effects of an increase in the degree of shock persistence are studied. We conclude that persistence in the macroseries is a natural feature of the dynamic path of economies such as ours.

Section 2 contains a formal model description and outlines our computational technique. Results are presented and reviewed in section 3, while section 4 provides a brief summary and conclusion.

2. The model

The computation to be outlined approximates the solution (derivation of the optimal consumption/savings policies) to the following 'central planning' problem:[1]

$$\max \mathrm{E}\left(\sum_{t=0}^{\infty} \beta^t U(c_t)\right),$$

$$\text{(P)} \qquad \text{s.t.} \quad c_t + k_{t+1} \leqq f(k_t) r_t = q_t,$$

$$H(r^*_{t+1}) = \int_{\underline{r}}^{r^*_{t+1}} \int_{\underline{r}}^{\bar{r}} \mathrm{d}G(r_{t+1}, r_t)\, \mathrm{d}H(r_t)\, \mathrm{d}r_{t+1},$$

where c_t, k_t, and q_t are, respectively, per capita consumption, capital stock, and output in period t. The concave production technology $f(\ )$ is shocked in each period by the random factor $\tilde{r}_t, \underline{r} \leqq \tilde{r}_t \leqq \bar{r}$, with stationary distribution function $H(.)$, and transition density function $\mathrm{d}G(.,.)$. Since the distribution of shocks in period $t+1$ is thus influenced by the state in period t, shocks are correlated through time. Lastly, $U(.)$ represents the (increasing, concave) period utility function, and β the societal discount (time preference) parameter.

We explicitly solve problem (P) above for the cases in which $U(c) = c^\gamma/\gamma$, $\gamma \in \{-2, -1, 0(\ln c), 0.25, 0.5\}$, $\beta \in \{0.8, 0.9, 0.95\}$, $f(k) = Ak^\alpha$, $A = 2/3$, $\alpha \in \{0.25, 0.5\}$, $r_t \in \{1.5, 1, 0.5\}$, and for which the transition density for $\tilde{r}_t$ is approximated as a 3×3 transition probability matrix $\{\Phi\}$, where the ϕ_{uv}th entry indicates the probability that $r_{t+1} = r_v$ given $r_t = r_u$. To understand the effects of shock correlation we solve problem (P) for a variety of symmetric matrices $\{\Phi\}$ where $\phi_{uu} = c$, $\phi_{uv} = 1 - c/2$, $u \neq v$. By allowing ϕ_{uu} to assume values from the set $\{0.333, 0.5, 0.6, 0.7, 0.8, 0.9\}$, we are able to model progressively greater shock correlation.

Our parameter choices (in particular the fact that $A = 2/3$) constrain the problem in such a way as to ensure that the range of each of the series of consumption, output, and capital stock lies in the interval $[0, 1]$. Choosing capital stock as the state variable, we define a partition $\Pi = \{k_i\}$ of this range, for which, for any two adjacent elements of the partition $k_i, k_{i+1}, k_{i+1} - k_1 = 0.01$. This partition then serves as the set of feasible states for capital stock. To determine the optimal savings and consumption policies for each of the parameter choices, we follow the customary dynamic programming method of seeking a fixed point to the related functional equation by a sequence of

[1] The reader is once again reminded that the policy functions arising from the solution to this central planning formulation coincide with the aggregate consumption and investment functions of the analogous homogeneous consumer economy in recursive competitive equilibrium.

approximating iterations. Under this procedure, the optimal savings policies $s(k_i, r_u) = (k_{t+1})$ are found as limits of sequences $s_n(k_i, r_u)$, where $s_n = s_n(.,.)$ solves

$$V^n(k_i, r_u) = \max\left\{u(f(k_i)r_u - s_n) + \beta \sum_{v=1}^{3} V^{n-1}(s_n, s_v)\phi_{uv}\right\},$$

$$s_n \in \Pi, \qquad s_n \leqq f(k_i)r_u.$$

In this formulation, the stationary savings policy function depends not only upon capital stock (or output), but also upon the shock realization because of the information it provides as to the distribution of next period's shock.

Given these savings functions, the transition probability matrix for the Markov process on capital stock can then be constructed. Given this matrix, the stationary distribution on capital stock (from which can be derived the analogous distributions on output and consumption) can be obtained by solving the associated linear programming problem. The time plots of the various series were obtained from simply recording the successive values of the savings function, for one-hundred values of the random factor chosen from a random number generator which represented the transition matrix under study.[2]

In a similar fashion, the aforementioned measures of autocorrelation of consumption, output, and capital stock and other statistical quantities were calculated from time plots of the economy over four-thousand periods for the various parameter choices. Computational results only for the cases in which $\phi_{uu} \in \{0.33, 0.5, 0.7, 0.9\}$ will be reported, however; the omitted cases offer identically parallel results.[3]

We have resorted to numerical computation as it is, in general, not possible to derive closed form solutions for the optimal policy functions and the resultant stationary distributions on capital stock, consumption and output. To illustrate, consider the derived (approximate) stationary distribution on capital stock for the parameter choices $\gamma = 0.5$, $\beta = 0.95$ and $\alpha = 0.25$ where $\phi_{uu} = 0.33$. In this case we find that for $k \in \{0.05, 0.08, 0.11\}$, $\text{Prob}(k) = 0.2222$, for $k \in \{0.06, 0.09, 0.10\}$, $\text{Prob}(k) = 0.1111$, and $\text{Prob}(k) = 0$ for k otherwise. It is seen that this distribution cannot be characterized by any well-known distribution function.

For a complete theoretical analysis of this model — conditions sufficient for the existence of policy functions and the resultant stationary states — the

[2]In each case, these hundred value series were chosen so as to be representative of the underlying distributions; for example, the initial values were always chosen to be elements of the corresponding stationary distributions.

[3]Complete documentation for the various computer programs is available from the authors.

reader is referred to Donaldson and Mehra (1983). Our choice of functional forms and persistence structures generally respects the assumptions of that paper.

3. Computation results

3.1. Effects of shock persistence on optimal policies

As noted by Donaldson and Mehra (1983), non-zero correlation between the random shocks implies that consumption and investment choices are not uniquely defined as functions of the output level. In fact, the current state of the economy is summarized by the capital stock (or output) level *and* the current realization of the random shock. The latter not only determines current productivity of inputs but also conditions the expectation of next period's random factor. Table 1 illustrates the effect of this modification on optimal behavior for the risk preference factor $\gamma=-2$. It shows the optimal savings levels corresponding to current capital stock (k_i column) for the different possible values of the random shock ($r_j=0.5$, 1, or 1.5). The left array is concerned with the i.i.d. case: the probability of obtaining any value for r next period is identical and independent of the current realization. The right array deals with the case where the probability of obtaining a repeated shock of identical value is 0.7.

Referring to table (1), shock persistence is seen to increase savings for the given values of capital stock and the random shock when the random realization is low ($r=0.5$), and to decrease savings when it is high ($r=1.5$). Thus for capital stock at a level $k=0.04$, the optimal savings for $r=0.5$, 1, and 1.5 are, respectively, 0.02, 0.08 and 0.15 under the i.i.d. hypothesis, and 0.03, 0.07, and 0.14 in the case of 0.7 persistence. Similarly, at the 0.28 capital stock level, optimal investment is, respectively, 0.06, 0.18, and 0.32 in the i.i.d. case, and 0.07, 0.17 and 0.29 in the other.

In order to interpret this result note first that an increase in the degree of shock persistence will effect a decrease in the expected rate of return on savings when the current realization of the exogenous shock is low, and an increase in the expected return when the shock value is high. Conversely, an increase in the degree of persistence will increase the variance of the rate of return on savings when the shock is low, and decrease the variance when it is high. [At a given savings level k, expected (gross) return on savings is $f'(\bar{k})\mathrm{E}r$, and the variance of the return on savings is $[f(\bar{k})]^2 \operatorname{var} r$. $\mathrm{E}r$ and $\operatorname{var} r$ clearly change as described above when shock correlation changes.] Now the impact of an increase in persistence can be interpreted as the sum of these two effects. The literature on the effect of an increase in risk (e.g. a mean preserving shift) on savings is abundant and clear: for the class of utility functions we consider, an increase in risk will increase, leave unchanged, or

Table 1

$\alpha=0.25, \beta=0.95, \gamma=-2.$

	$\phi_{uu}=0.333$ r_u			$\phi_{uu}=0.7$ r_u		
k_i	0.50	1.00	1.50	0.50	1.00	1.50
000	0.000	0.000	0.000	0.00000	0.00000	0.00000
010	0.010	0.040	0.090	0.02000	0.04000	0.08000
020	0.020	0.060	0.120	0.02000	0.06000	0.10000
030	0.020	0.070	0.140	0.03000	0.07000	0.12000
040	0.020	0.080	0.150	0.03000	0.07000	0.14000
050	0.030	0.090	0.170	0.03000	0.08000	0.15000
060	0.030	0.090	0.180	0.04000	0.09000	0.16000
070	0.030	0.100	0.190	0.04000	0.09000	0.17000
080	0.030	0.110	0.200	0.04000	0.10000	0.18000
090	0.030	0.110	0.210	0.04000	0.11000	0.19000
100	0.030	0.120	0.220	0.04000	0.11000	0.20000
110	0.040	0.120	0.230	0.05000	0.12000	0.20000
120	0.040	0.130	0.230	0.05000	0.12000	0.21000
130	0.040	0.130	0.240	0.05000	0.12000	0.22000
140	0.040	0.130	0.250	0.05000	0.13000	0.23000
150	0.040	0.140	0.260	0.05000	0.13000	0.24000
160	0.040	0.140	0.270	0.06000	0.13000	0.24000
170	0.040	0.140	0.270	0.06000	0.14000	0.25000
180	0.050	0.150	0.280	0.06000	0.14000	0.25000
190	0.050	0.150	0.280	0.06000	0.14000	0.26000
200	0.050	0.150	0.290	0.06000	0.15000	0.26000
210	0.050	0.160	0.290	0.06000	0.15000	0.27000
220	0.050	0.160	0.300	0.06000	0.15000	0.27000
230	0.050	0.160	0.300	0.07000	0.16000	0.27000
240	0.050	0.160	0.310	0.07000	0.16000	0.28000
250	0.050	0.170	0.310	0.07000	0.16000	0.28000
260	0.060	0.170	0.320	0.07000	0.16000	0.29000
270	0.060	0.180	0.320	0.07000	0.17000	0.29000
280	0.060	0.180	0.320	0.07000	0.17000	0.29000
290	0.060	0.180	0.330	0.07000	0.17000	0.30000
300	0.060	0.180	0.330	0.07000	0.17000	0.30000
310	0.060	0.190	0.340	0.07000	0.18000	0.30000
320	0.060	0.190	0.340	0.07000	0.18000	0.31000
330	0.060	0.190	0.340	0.07000	0.18000	0.31000
340	0.060	0.190	0.350	0.08000	0.18000	0.32000
350	0.060	0.190	0.350	0.08000	0.18000	0.32000
360	0.060	0.190	0.350	0.08000	0.18000	0.32000

decrease savings if, respectively, $\gamma<0$, $\gamma=0$ or $\gamma>0$ [e.g. Rothschild–Stiglitz (1971, p. 70)]. Although we have been unable to find an explicit analysis of the effect of an increase in expected return on savings [Sandmo (1974) asserts such a change will have 'no clear-cut effect, even in the simple case with one asset', but rather depend crucially upon the shape of the utility functions under consideration], we believe an analogous result holds for the constant relative risk aversion family of utility functions.[4] That is, an increase in expected rate of return on savings will decrease, leave unchanged, or increase savings if, respectively, $\gamma<0$, $\gamma=0$ or $\gamma>0$.

These conclusions, together with our earlier observations on the effect of an increase in the degree of persistence provide a complete intuitive explanation for the results of table 1: Going from $\phi_{uu}=0.333$ to $\phi_{uu}=0.7$ corresponds, for the $r=1.5$ column, to an increase in expected value and a decrease in variance of returns. With $\gamma=-2$, these two changes reinforce one another to generate a decrease in savings; conversely for the 0.5 column. Furthermore, we would now expect that no change in optimal policies would be recorded for the $\gamma=0$ case. This is, indeed, what is observed [Long and Plosser (1980, p. 29) make an analogous observation for their model]. Finally, as expected, the opposite effect of increasing the degree of shock persistence on savings is observed when γ is positive.

An implication of these results is that for $\gamma<0$ an increase in persistence reduces the range of the equilibrium capital stock distribution, while increasing the range for $\gamma>0$. To illustrate, we see from table 1 that the range goes from [0.02, 0.35] in the 0.333 case to [0.03, 0.30] for the correlated 0.7 case.

3.2. Impact of persistence on optimal time paths

To illustrate the effects resulting from increased correlation in the random shock, we first calculated the means, variances, and coefficients of variation for the stationary distributions of aggregate consumption, output, and capital stock. This information is provided for $\phi_{uu}\in\{0.333, 0.5, 0.7, 0.9\}$ in tables 2, 3 and 4 (to which our subsequent comments will refer).

Let us first attempt to interpret the $\gamma=0$ (table 2) case. Indeed, we know that for $\gamma=0$, optimal policies are not affected by increased shock persistence. The recorded changes in the moments are thus the direct result of the change in correlation. We observe that the variance of the three series consistently rises as correlation is increased. This 'spreading out' of the three distributions may be explained by realizing that the longer the series of favorable ($r=1.5$) shocks, the higher the output rises (until the economy reaches the upper bound of the output range). Similarly for the other two series. Conversely,

[4]To illustrate, take the analogous two-period model,

$$\max_{c_t} \mathrm{E}\left[(c_t)^{\gamma}\frac{1}{\gamma}+\beta(c_{t+1})^{\gamma}\frac{1}{\gamma}\right] \quad \text{s.t.} \quad c_{t+1}=(q_t-c_t)(1+\tilde{r}_t).$$

The first-order condition is

$$[c_t/(q_t-c_t)]^{\gamma-1}=\beta\,\mathrm{E}(1+\tilde{r}_t)^{\gamma},$$

which clearly confirms the above observation.

Table 2

$\gamma=0.$

ϕ_{uu}		Capital	Output	Consumption
0.33	E	0.0842	0.3527	0.2685
	var	0.0013	0.0225	0.0131
	coeff. var.	0.4228	0.4256	0.4270
0.5	E	0.0861	0.3583	0.2722
	var	0.0014	0.0256	0.0149
	coeff. var.	0.4405	0.4468	0.4492
0.7	E	0.0858	0.3616	0.2758
	var	0.0018	0.0300	0.0171
	coeff. var.	0.4985	0.4793	0.4737
0.9	E	0.0886	0.3714	0.2828
	var	0.0022	0.0354	0.0200
	coeff. var.	0.5279	0.5064	0.4999

Table 3

$\gamma=0.25.$

ϕ_{uu}		Capital	Output	Consumption
0.33	E	0.0826	0.3530	0.2705
	var	0.0008	0.0219	0.0141
	coeff. var.	0.3600	0.4137	0.4387
0.5	E	0.0829	0.3565	0.2735
	var	0.0010	0.0245	0.0156
	coeff. var.	0.3850	0.4395	0.4565
0.7	E	0.0862	0.3635	0.2773
	var	0.0014	0.0290	0.0177
	coeff. var.	0.4342	0.4683	0.4793
0.9	E	0.0878	0.3700	0.2822
	var	0.0016	0.0333	0.0203
	coeff. var.	0.4582	0.4932	0.5043

Table 4.

$\gamma=-2.$

ϕ_{uu}		Capital	Output	Consumption
0.33	E	0.1272	0.3772	0.2500
	var	0.0080	0.0302	0.0073
	coeff. var.	0.7033	0.4607	0.3416
0.5	E	0.1310	0.3865	0.2555
	var	0.0086	0.0360	0.0096
	coeff. var.	0.7093	0.4909	0.3832
0.7	E	0.1342	0.3987	0.2645
	var	0.0087	0.0427	0.0131
	coeff. var.	0.6966	0.5182	0.4325
0.9	E	0.1155	0.3920	0.2765
	var	0.0060	0.0451	0.0184
	coeff. var.	0.6691	0.5416	0.4910

the longer the series of low shocks, the closer the economy approaches the lower bounds of the relevant ranges. Clearly, the greater the degree of shock correlation, the longer a series of bad (or good) realizations can persist.

To this 'direct' effect of increased correlation, we must add, when $\gamma \neq 0$, the impact of this change on optimal policies. When shock correlation increases and $\gamma > 0$, savings is increased for high realizations of r, decreased for low realizations. This tends to reinforce the original effect of increased correlation, and we would thus expect the pattern of variances increasing with the degree of shock persistence to be maintained for all three aggregate series. This indeed is reflected in the results reported in table 3. Conversely, with $\gamma < 0$, the change in optimal policies tends to weaken the original 'direct' effect of the change in shock correlation, so that no clear-cut statements can be made. Indeed, table 4 shows that while the variances of output and consumption continue to monotonically increase with increases in the degree of shock correlation, no similarly monotonic pattern is observed for the capital stock. [The variances for the 0.9 case and the (non-reported) 0.8 case are 0.0060 and 0.0077, respectively, which are lower than the variance for the 0.333 case.] Except for the $\gamma = -2$ capital stock series, the coefficient of variation always increases with an increase in the degree of persistence showing that the effects on the standard deviations are always more pronounced than those on the means. We note, however, that generally the mean of a series tends to move together with its variance.

In summary, we can say that if we choose to measure economic variation by the variance of the stationary distribution of consumption or output, increased correlation (suggesting more precise knowledge of the future) results in greater variability (as will be discussed shortly, however, the opposite effect is observed if period to period variation is measured). Notice also that for every shock structure the corresponding variance of consumption declines as γ declines while the variance of output simultaneously rises. This confirms observations first made in Danthine and Donaldson (1981) — reductions in consumption risk may be effected only at the cost of higher output variability.

We now consider the effects of shock persistence on period to period output correlation (similar effects will be observed for capital stock and consumption). The representative tables 5 and 6 describe the autocorrelation of q_t with $q_{t-\theta}$ for $\theta = 1, 2, 3$.

As would be expected, the effect of increased shock persistence is, in all cases, to increase output correlation. It is also observed that for all levels of shock persistence output correlation is greater as the agent becomes more risk-averse. Again, this is not surprising. More risk-averse agents ($\gamma = -2$), desiring to reduce consumption variability, will adapt their investment behavior so as to make future output (and thus consumption) as similar as possible to today's thereby *increasing* output persistence. These effects are

Table 5

$\alpha=0.25, \beta=0.95, \gamma=-2.$

	Autocorrelation, $q_t, q_{t-\theta}$ ϕ_{uu}			
	0.333	0.5	0.7	0.9
$\theta=1$	0.43	0.60	0.76	0.92
$\theta=2$	0.19	0.31	0.52	0.82
$\theta=3$	0.11	0.17	0.32	0.73

Table 6

$\alpha=0.25, \beta=0.95, \gamma=0.5.$

	Autocorrelation, $q_t, q_{t-\theta}$ ϕ_{uu}			
	0.333	0.5	0.7	0.9
$\theta=1$	0.18	0.43	0.69	0.89
$\theta=2$	0.04	0.14	0.42	0.76
$\theta=3$	0.03	0.04	0.24	0.64

confirmed for positive values of γ. Autocorrelation results ($\gamma=-2$) for consumption and capital stock are closely similar to those of output.

A graphical illustration of these results is provided in figs. 1–4, where we have plotted the time path of output for one-hundred periods for transition matrices with representative ϕ_{uu} elements once again chosen from the set $\{0.333, 0.5, 0.7, 0.9\}$.

The persistence in the random shock is clearly manifest as persistence in the output series. Note also that considerable persistence is evidenced even in the i.i.d. ($\phi_{uu}=0.333$) case (recall from tables 4 and 5 that autocorrelation is non-zero). This latter effect may be explained by noting that in this model, for the i.i.d. case, savings is an increasing function of output alone [see Brock and Mirman (1972) or Danthine and Donaldson (1981) for details] and thus the effect of a high realization (high output) is manifest the next period in the form of a higher capital stock level. Analogously, a low realization this period suggests low saving and low capital stock next period and hence low output next period. Consequently, even in the i.i.d. case, there is a tendency for the output series to evidence persistence through time. This same effect is observed by Long and Plosser (1980, p. 20 and elsewhere).

This effect is further magnified by persistence in the random shock. As shown in Donaldson and Mehra (1983), savings $s=s(f(k_t, r_t), r_t)$ is an increasing function not only of capital stock (output) but of the shock r_t as

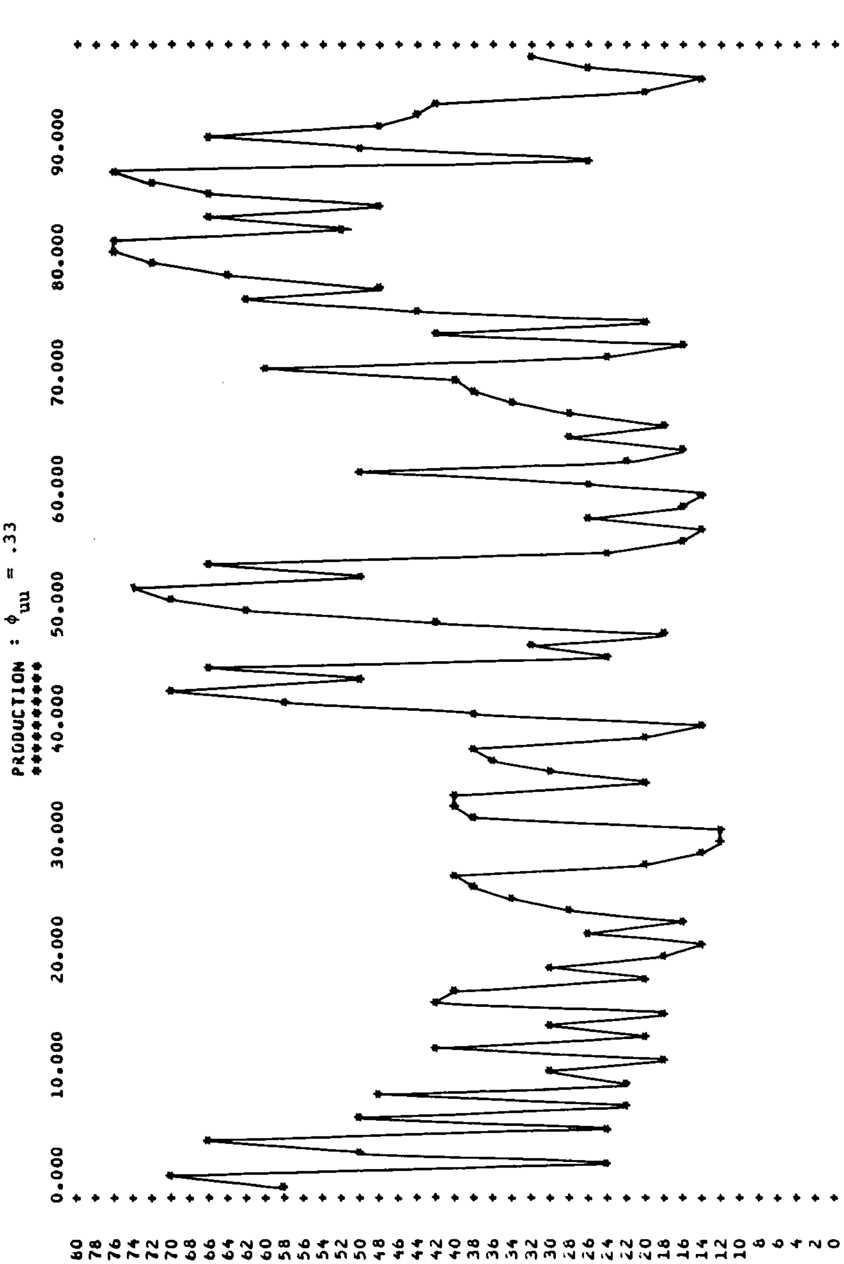
PRODUCTION : ϕ_{uu} = .33
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
80 78 76 74 72 70 68 66 64 62 60 58 56 54 52 50 48 46 44 42 40 38 36 34 32 30 28 26 24 22 20 18 16 14 12 10 8 6 4 2 0

Fig. 1

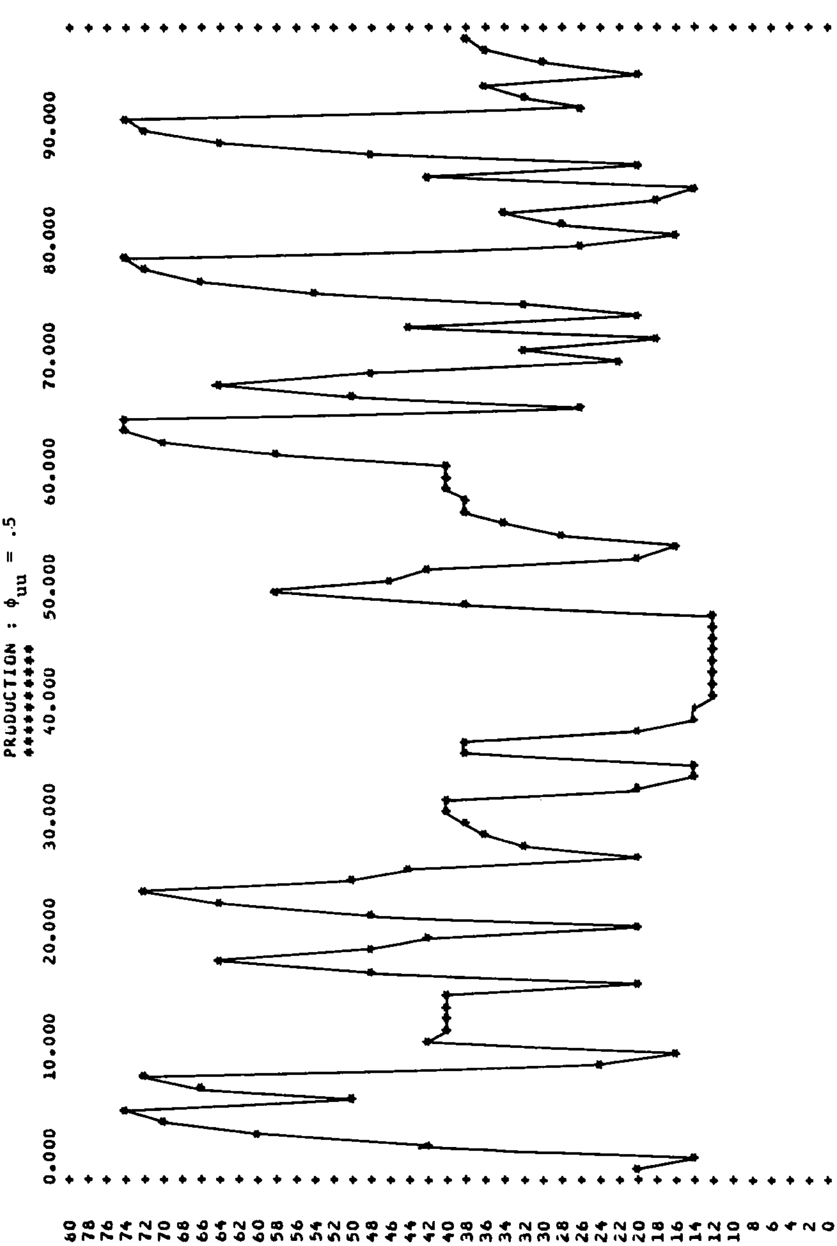
PRODUCTION : ϕ_{uu} = .5
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
80 78 76 74 72 70 68 66 64 62 60 58 56 54 52 50 48 46 44 42 40 38 36 34 32 30 28 26 24 22 20 18 16 14 12 10 8 6 4 2 0

Fig. 2

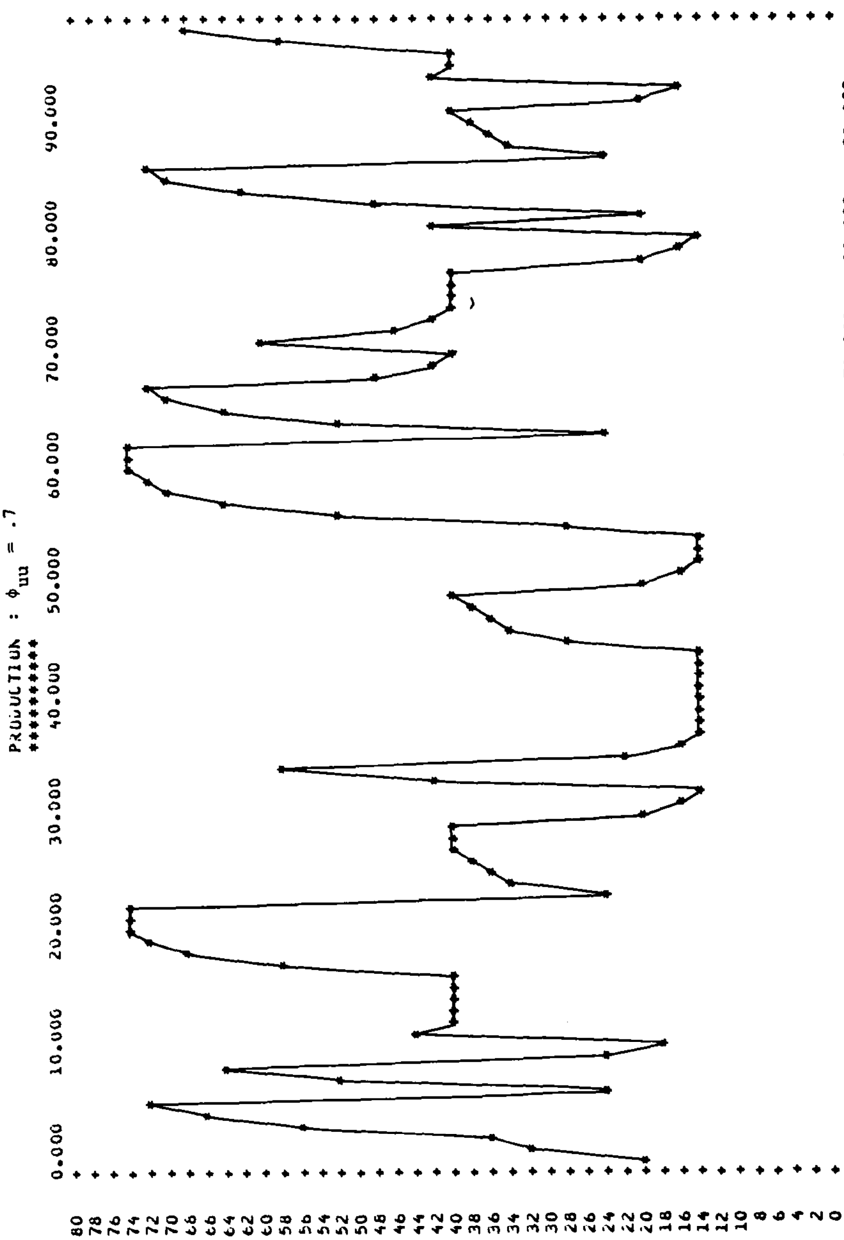
PRODUCTION : ϕ_{uu} = .7
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
80 78 76 74 72 70 68 66 64 62 60 58 56 54 52 50 48 46 44 42 40 38 36 34 32 30 28 26 24 22 20 18 16 14 12 10 8 6 4 2 0

Fig. 3

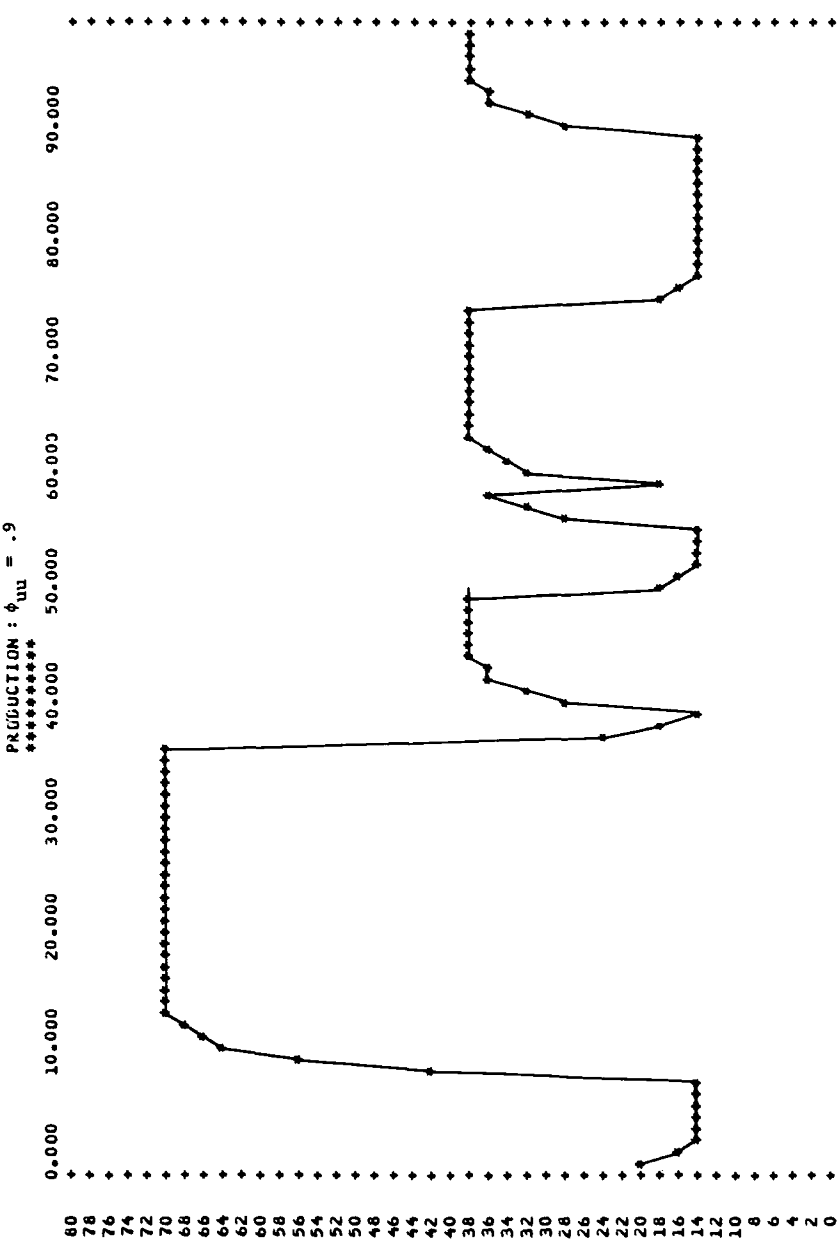
PRODUCTION : ϕ_{uu} = .9
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
80 78 76 74 72 70 68 66 64 62 60 58 56 54 52 50 48 46 44 42 40 38 36 34 32 30 28 26 24 22 20 18 16 14 12 10 8 6 4 2 0

Fig. 4

well. Hence the above effect is exaggerated. Similar persistence is observed in the aggregate series of consumption and capital stock. We illustrate these effects on the consumption series for the cases $\phi_{uu}=0.5$ and $\phi_{uu}=0.9$ in figs. 5 and 6.

4. Concluding comments

In this note we have sought to understand the impact of correlation in the economy's random productivity shock on its optimal policies and resulting steady state. Although the impact on optimal policies differs according to the degree of concavity of the utility function, we observe that an increase in shock persistence (increasing the probability that 'tomorrow will closely resemble today') introduces, somewhat paradoxically, greater variability (as measured by the stationary variance of the relevant series) in the macroaggregates (except for capital stock when $\gamma=-2$). We next sought to illustrate, and to provide some measure of how persistence in exogenous shocks translated into persistence in the three macroseries. It was observed that persistence exists in the capital, output, and consumption series even when shocks are not serially correlated. In this sense persistence is a natural feature of the dynamics of the macro aggregates in an economy such as ours. Furthermore, once persistence on aggregate shocks is introduced, the degree of autocorrelation in our different series rapidly reaches the order of magnitude of measures of autocorrelation computed from real world data [see Hodrick and Prescott (1980)].[5]

This study could in principle lend itself to two interesting questions. First, it would be useful to develop a more rigorous procedure to test the model than the one consisting in comparing correlation coefficients with those obtained in real data. At least, if we quadratically approximate our objective function, it would seem possible to estimate preference and technology parameters using a maximum likelihood procedure in the spirit of Hansen–Sargent (1980). Second, we could systematically examine whether other stylized properties of business cycles can be accounted for in such a model [see Lucas (1977) for a description]. It can be observed, for example, that the three aggregate series move together: contemporaneous correlation coefficients are of the order of 0.95 [compared with an order of magnitude of 0.9 as reported by Hodrick–Prescott (1980) for actual U.S. data for the second half of their 1950–1979 sample].[6] We could also compute implicit

[5]Other effects were studied beyond those reported in this paper. Some, such as the influence of persistence on the proportions of output consumed and invested, were omitted from the discussion because no systematic effects were observed. Others, such as the effects of an increase in β, were omitted as they had been analyzed elsewhere with identical results [see Danthine and Donaldson (1981)].

[6]This is not surprising since our consumption and savings policies are approximately linear in output.

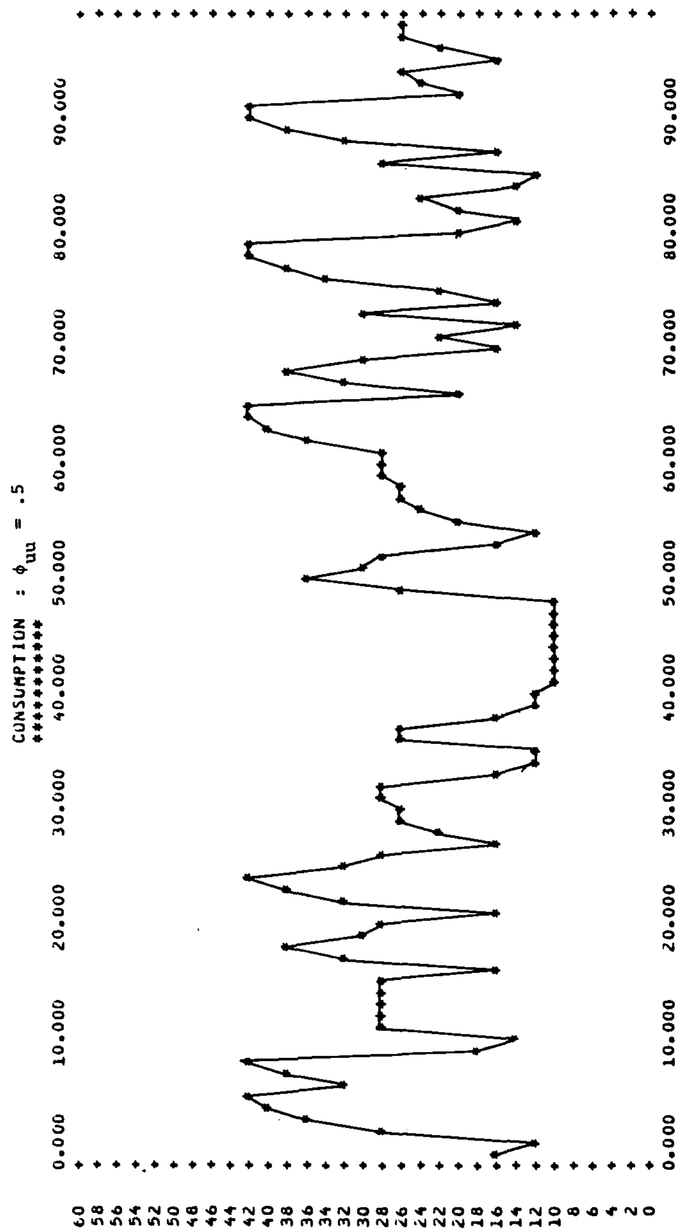
CONSUMPTION : ϕ_{uu} = .5
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
60
58
56
54
52
50
48
46
44
42
40
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0

Fig. 5

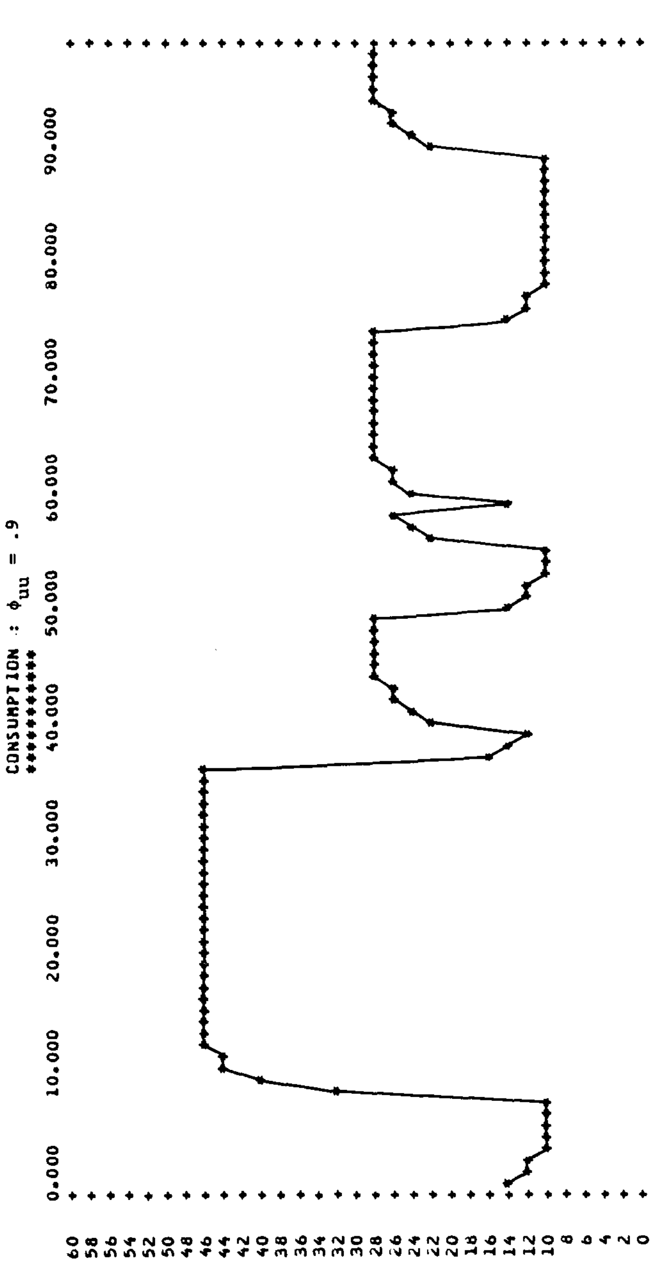
CONSUMPTION : ϕ_{uu} = .9
0.000
10.000
20.000
30.000
40.000
50.000
60.000
70.000
80.000
90.000
60
58
56
54
52
50
48
46
44
42
40
38
36
34
32
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0

Fig. 6

interest rates and study the co-movements of 'market' and 'risk-free' returns. However, we feel that as it stands the model is too simple and aggregated to permit a more elaborate discussion or justify sizeable additional investment. Further work will pursue these two objectives on the basis of a model allowing for a distinction between investment and capital stock, and an explicit labor–leisure choice.

Appendix

In this appendix, we work out a parametric example of our model. This example is the only one we are aware of for which closed form solutions are available. It provides confirmation of the results of section 3.2 concerning expected values of the different variables. Confirmation of the variance results of that section appears prohibitively difficult to obtain.

Let

$$u(c)=\ln c, \qquad f(k)r=k^{\alpha}r,$$

$$\ln r_{t+1}=a+\rho\ln r_t+\varepsilon_t, \qquad \text{or}$$

$$r_{t+1}=\mathscr{L}\, r_t^{\rho}\, \mathrm{e}^{\tilde{\varepsilon}_t}, \qquad \tilde{\varepsilon}\sim \text{i.i.d.}, \qquad \mathcal{N}(0;\sigma^2).$$

As suggested in the text [see also Donaldson and Mehra (1982)], the policy functions are not altered by the persistence in the shocks. They are

$$k_{t+1}=\alpha\beta k_t^{\alpha} r_t, \tag{1}$$

$$c_t=(1-\alpha\beta)k_t^{\alpha} r_t, \tag{2}$$

as in the case where the r's are i.i.d. [see Danthine–Donaldson (1981)]. Used repetitively, (1) implies

$$k_t=(\alpha\beta)^{\sum_{s=0}^{t-1}\alpha^s} k_0^{\alpha^t}\prod_{s=0}^{t-1} r_s^{\alpha^{t-1-s}}. \tag{3}$$

Furthermore,

$$\prod_{s=0}^{t-1} r_s^{\alpha^{t-1-s}}=\mathscr{L}^{\sum_{s=0}^{t-2}\alpha^s\sum_{\tau=0}^{t-2}\rho^{\tau}}\, r_0^{\sum_{s=0}^{t-1}\rho^s\alpha^{t-s-1}}\, \mathrm{e}^{\sum_{s=0}^{t-2}\alpha^s\sum_{\tau=0}^{t-s-2}\rho^{t-s-2-\tau}\varepsilon_\tau}. \tag{4}$$

Using (4) into (3), we can express $\lim_{t\to\infty} \mathrm{E}k_t$ as

$$\lim_{t\to\infty} \mathrm{E}k_t=\alpha\beta^{1/(1-\alpha)}\mathscr{L} \lim_{t\to\infty} \mathrm{E}\big[\mathrm{e}^{\sum_{s=0}^{t-2}\alpha^s\sum_{\tau=0}^{t-s-2}\rho^{t-s-2-\tau}\varepsilon_\tau}\big], \tag{5}$$

where

$$\tilde{\mathscr{L}} = \lim_{t \to \infty} \mathscr{L}^{\sum_{s=0}^{t-2} \alpha^s \sum_{\tau=0}^{t-2-s} \rho^\tau};$$

this limit exists as, for $t \to \infty$, the exponent is a (monotonically) increasing sequence of positive numbers and this sequence is bounded above by $(1/(1-\rho))(1/(1-\alpha))$.

In (5), the term in r_0 has disappeared, this is because

$$\sum_{s=0}^{t-1} \rho^s \alpha^{t-s-1} \to 0 \quad \text{as} \quad t \to \infty.$$

This can be seen as follows: let $\delta = \max[\alpha, \rho]$. Then

$$\sum_{s=0}^{t-2} \rho^s \alpha^{t-s-1} \leqq t\delta^{t-1}.$$

In order to be sure that $t\delta^{t-1} \to 0$ as $t \to \infty$, it is sufficient to show that $\sum_{t=1}^{\infty} a_t = \sum_{t=1}^{\infty} t\delta^{t-1}$ converges (all terms are positive). This follows from the Ratio Test:

$$\lim_{t \to \infty} |a_{t+1}/a_t| = \lim_{t \to \infty} |(t+1)\,\delta^t / t\,\delta^{t-1}| = \delta < 1.$$

Now, since we are interested in the sign of $\partial \lim_{t \to \infty} \mathrm{E}k_t / \partial \rho$, we are simply going to show that the three components in the R.H.S. of (5) have positive or zero derivatives w.r.t. ρ. This is obvious for the two first terms. Regarding the term inside the expectation operator, let us use the exponential approximation

$$\mathrm{e}^x = x + x^2/2! + x^3/3! + \ldots$$

That term can thus be written as

$$\sum_{s=0}^{t-2} \alpha^s \sum_{\tau=0}^{t-s-2} \rho^{t-s-2-\tau} \varepsilon_\tau + \frac{1}{2!} \left(\sum_{s=0}^{t-2} \alpha^s \sum_{\tau=0}^{t-s-2} \rho^{t-s-2-\tau} \varepsilon_\tau \right)^2 + \ldots$$

Using the expectation operator, the first term vanishes since $\mathrm{E}\varepsilon_\tau = 0, \forall \tau$. Furthermore the second term will be a sum of elements of the form

$$\alpha^x \rho^y \,\mathrm{E}\varepsilon_\tau \varepsilon_\theta.$$

This is zero if $\tau \neq \theta$ as ε's are i.i.d.; it is $\alpha^x \rho^y \sigma^2$ if $\tau = \theta$. In this case, it increases with ρ. Thus

$$\frac{\partial}{\partial \rho} \mathrm{E}\left[\frac{1}{2}\sum_{s=0}^{t-2} \alpha^s \sum_{\tau=0}^{t-s-2} \rho^{t-s-2-\tau} \varepsilon_\tau\right]^2 > 0.$$

By the same reasoning and observing that all odd moments of ε are zero, the same is true for the other terms of the exponential approximation.

Thus, for the example at hand, the asymptotic mean of the capital stock increases with ρ. The same is true for consumption, as is clear from (2), and consequently for output. These results all confirm those reported in table 2 with ρ in this example playing the role of ϕ_{uu} in the text.

References

Brock, William A., 1978, An integration of stochastic growth theory and the theory of finance, Department of Economics working paper (University of Chicago, Chicago, IL).

Brock, William A. and Leonard J. Mirman, 1972, Optimal economic growth and uncertainty: The discounted case, Journal of Economic Theory 4, 497–513.

Cass, David, 1965, Optimum growth in an aggregative model of capital accumulation, Review of Economic Studies 32, 233–240.

Danthine, Jean-Pierre and John B. Donaldson, 1981, Stochastic properties of fast vs. slow growing economies, Econometrica 49, 1007–1033.

Donaldson, John B. and Rajnish Mehra, 1982, Comparative dynamics of equilibrium intertemporal asset pricing model, Working paper (Columbia University, New York).

Donaldson, John B. and Rajnish Mehra, 1983, Stochastic growth with correlated production stocks, Journal of Economic Theory, forthcoming.

Hansen, Lars and Thomas Sargent, 1980, Formulating and estimating dynamic linear rational expectations systems, Journal of Economic Dynamics and Control 2, 7–46.

Hodrick, Robert J. and Edward C. Prescott, 1980, Post war U.S. business cycles: An empirical investigation, Working paper (Northwestern University, Evanston, IL).

Koopmans, Tjalling, 1965, On the concept of optimal economic growth, in: Semaine d'étude sur le rôle de l'analyse èconometrique dans la formulation de plans de développement, Pontifican Academiae Scientiarium Scripta Varia, No. 28 (Vatican, Rome).

Kydland, Finn and Edward C. Prescott, 1982, Time to build and the persistence of unemployment, Econometrica 50, 1345–1370.

Long, John B. and Charles L. Plosser, 1980, Real business cycles, Working paper (University of Rochester, Rochester, NY).

Lucas, Robert E. Jr., 1975, An equilibrium model of the business cycle, Journal of Political Economy 83, 1113–1144.

Lucas, Robert E. Jr., 1977, Understanding business cycles, in: Karl Brunner and Allan H. Meltzer, eds., Stabilization of the domestic and international economy, Vol. 5 of the Carnegie–Rochester Series on Public Policy (North–Holland, Amsterdam) 7–29.

Prescott, Edward C. and Rajnish Mehra, 1980, Recursive competitive equilibrium: The case of homogeneous households, Econometrica 48, 1365–1379.

Rothschild, Michael and Joseph E. Stiglitz, 1971, Increasing risk II: Its economic consequences, Journal of Economic Theory 4, 66–84.

Sandmo, Agnar, 1974, Two-period models of consumption and decision under uncertainty: A survey in: J.H. Drèze, ed., Allocation under uncertainty equilibrium and optimality (Wiley, New York).